Teaching Mathematics: Big Ideas to Promote Understanding

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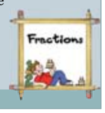


Road Map to Presentation

- Overview of foundational documents
  - Curriculum Focal Points, National Council of Teachers of Mathematics, 2006
  - Foundations for Success, The National Mathematics Advisory Panel Final Report, 2008
  - Common Core State Standards in Mathematics (2010)
- Mathematics Difficulties
  - Need for explicit, systematic instruction
- Whole Number arithmetic: Big Ideas
  - IES Practice Guide, Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools
- Fractions: Big Ideas
  - IES Practice Guide: Developing Effective Fractions Instruction for Kindergarten Through 8th Grade
- Summary

Curriculum Focal Points *for* Prekindergarten *through* Grade 8 Mathematics

5

Curriculum Focal Points for Mathematics in Prekindergarten through Grade 8

A Quest for Coherence

Three curriculum focal points are identified and described for each grade level, pre-K–8, along with connections to guide integration of the focal points at that grade level and across grade levels, to form a comprehensive mathematics curriculum. To build students' strength in the use of mathematical processes, instruction in these content areas should incorporate—

- the use of the mathematics to solve problems;
- an application of logical reasoning to justify procedures and solutions; and
- an involvement in the design and analysis of multiple representations to learn, make connections among, and communicate about the ideas within and outside of mathematics.

The purpose of identifying these grade-level curriculum focal points and connections is to enable students to learn the content in the context of a focused and cohesive curriculum that implements problem solving, reasoning, and critical thinking.

These curriculum focal points should be considered as major instructional goals and desirable learning expectations, not as a list of objectives for students to master. They should be implemented with the intention of building mathematical competency for all students, bolstered by the pedagogical understanding that not every student learns at the same rate or acquires concepts and skills at the same time.

Those who are involved in curriculum planning for grades 6–8 should note that this set of curriculum focal points has been designed with the intention of providing a three-year middle school program that includes a full year of general mathematics in each of grades 6, 7, and 8. Those whose programs offer an algebra course in grade 8 (or earlier) should consider including the curriculum focal points that this framework calls for in grade 8 in grade 6 or grade 7. Alternatively, these topics could be incorporated into the high school program. Either way, curricula would not omit the important content that the grade 7 and grade 8 focal points offer students in preparation for algebra and for their long-term mathematical knowledge.

NCTM

Success

The Final Report of the National Mathematics Advisory Panel

2008

U.S. Department of Educati

Table 2: Benchmarks for the Critical Foundations

Fluency With Whole Numbers

1) By the end of Grade 3, students should be proficient with the addition and subtraction of whole numbers.
2) By the end of Grade 5, students should be proficient with multiplication and division of whole numbers.

Fluency With Fractions

1) By the end of Grade 4, students should be able to identify and represent fractions and decimals, and compare them on a number line or with other common representations of fractions and decimals.
2) By the end of Grade 5, students should be proficient with comparing fractions and decimals and common percent, and with the addition and subtraction of fractions and decimals.
3) By the end of Grade 6, students should be proficient with multiplication and division of fractions and decimals.
4) By the end of Grade 6, students should be proficient with all operations involving positive and negative integers.
5) By the end of Grade 7, students should be proficient with all operations involving positive and negative fractions.
6) By the end of Grade 7, students should be able to solve problems involving percent, ratio, and rate and extend this work to proportionality.

Geometry and Measurement

1) By the end of Grade 5, students should be able to solve problems involving perimeter and area of triangles and all quadrilaterals having at least one pair of parallel sides (i.e., trapezoids).
2) By the end of Grade 6, students should be able to analyze the properties of two-dimensional shapes and solve problems involving perimeter and area, and analyze the properties of three-dimensional shapes and solve problems involving surface area and volume.
3) By the end of Grade 7, students should be familiar with the relationship between similar triangles and the concept of the slope of a line.

COMMON CORE STATE STANDARDS FOR Mathematics

- The standards
  - define what students should understand and be able to do in their study of mathematics
  - set grade-specific standards but do not define the intervention methods or materials necessary to support students who are well below or well above grade-level expectations

CCSS for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Students with Mathematics Difficulties (MD)

- Problems in many areas of mathematics:
  - Inefficient retrieval of basic arithmetic combinations (Jordan, Hanich & Kaplan, 2003)
  - Delayed adoption of efficient counting strategies
  - Very limited working memory (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; McLean & Hitch, 1999; Swanson & Sachse-Lee, 2001)
  - Difficulties in many aspects of basic number sense (Jordan, Hanich, & Kaplan, 2003; Fuchs, Compton, Fuchs, Paulsen, Bryant, & Hamlett, 2005)
  - Problems with attention span and task persistence (DiPerna, Lei & Reid, 2007; Fuchs, 2005)

(National Mathematics Advisory Panel, 2008)

Recommendation	Level of evidence
Tier 1	
1. Screen all students to identify those at risk for potential mathematics difficulties and provide interventions to students identified as at risk.	Moderate
Tiers 2 and 3	
2. Instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers in kindergarten through grade 5 and on rational numbers in grades 4 through 8. These materials should be selected by committee.	Low
3. Instruction during the intervention should be explicit and systematic. This includes providing models of proficient problem solving, verbalization of thought processes, guided practice, corrective feedback, and frequent cumulative review.	Strong
4. Interventions should include instruction on solving word problems that is based on common underlying structures.	Strong
5. Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.	Moderate
6. Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.	Moderate
7. Monitor the progress of students receiving supplemental instruction and other students who are at risk.	Low
8. Include motivational strategies in tier 2 and tier 3 interventions.	Low

Source: Authors' compilation based on analysis described in text.

Discovery Learning

Children actively construct their own solutions to mathematics problems

- True discovery is rare and inefficient
- Impossible for students to discover <u>all</u> they need to know
- Discovery learning can be full of errors

Harris & Pressley (1991); Mayer (2004)

Direct Instruction

- Structured, systematic instruction that is teacher directed and includes opportunities for practice with corrective feedback.

(Stein, Silbert, Carnine, & Kinder, 2005)

Teaching students at-risk for mathematics difficulties

- Instruction should not be based on extreme positions
- Decisions about how to help students reach learning goals can never be made with absolute certainty

National Research Council (2001)

Explicit, Systematic Instruction

- Provide models of proficient problem solving
- Verbalize thought processes
- Guided practice
- Corrective feedback
- Frequent cumulative review

Gersten, Beckmann, et al. (2009)

Conspicuous Strategies

Expert actions for problem solving are made overt through teacher and peer modeling.

Teach
- the general case for which the strategy works.
- both how and when to apply the strategy.
- when the strategy does not work.

Coyne, Kameenui, & Carnine (2007)

Instructional Scaffolding

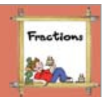
Provide strategic support at critical points during instruction

- Task Scaffolds:
  - Introduce concepts and skills systematically in increasing levels of difficulty
  - Sequence instruction to avoid confusion of similar concepts
  - Carefully select and sequence examples to reinforce learned material
  - Pre-teach prerequisite knowledge
- Material scaffolds:
  - Include visual prompts (e.g., graphic organizer), procedural facilitators, and manipulative materials

Coyne, Kameenui, & Carnine (2007)

Opportunities for Practice with High Quality Feedback

- Provide students with
  - multiple opportunities to practice with immediate high-quality feedback.
  - systematic review that is carefully scheduled and distributed to consolidate and integrate learning
  - high quality feedback that is immediate, individualized, and content specific.



Big Ideas in …

WHOLE NUMBERS

FRACTIONS

Whole number arithmetic

- Foundational to whole number arithmetic includes
  - Counting and cardinality
  - Basic facts
  - Base-ten computations
  - Place value
  - Properties of arithmetic (commutative, associative, and distributive laws for addition and multiplication)
  - Standard algorithms for operations

Teaching for understanding (e.g., meanings and properties of operations) should occur at all levels: concrete, semi-concrete (representational), and abstract

CCSS (2010) and NMAP (2008)

Basic Facts

"Knowing basic number combinations – the single digit addition and multiplication pairs and their counterparts for subtraction and division - is essential" (NCTM, Principles and Standards, 2000, p. 32)

Basic Facts

- Basic facts are simple closed number sentences used in computation. They must be *understood* and memorized; these are the tools of computation
  - Addition and subtraction facts involve two one-digit addends
  - Multiplication and division facts involve two one-digit factors
- Once understanding and basic facts are mastered, the specific operation can be expanded readily using *place value*

Teaching Basic Facts

- Phase 1: Counting Strategies – object counting (e.g., blocks or fingers) or verbal counting to determine the answer
- Phase 2: Reasoning strategies – using known information to logically determine an unknown combination
  - If you did not know how much 8 + 5 was, how could you find out?
- Phase 3: Mastery –Efficient (fast and accurate) production of answers

Baroody (2006, p. 22)

	Addition	Subtraction
Counting	Direct modeling (counting objects and fingers) • Counting all • Counting on from first • Counting on from larger	Counting objects • Separating from • Separating to • Adding on
	Counting abstractly • Counting all • Counting on from first • Counting on from larger	Counting fingers • Counting down • Counting up Counting abstractly • Counting down • Counting up
Reasoning	Properties • $a + 0 = a$ • $a + 1$ = next whole number • Commutative property	Properties • $a - 0 = a$ • $a - 1$ = previous whole number
	Known-fact derivations (e.g., $5 + 6 = 5 + 5 + 1$; $7 + 6 = 7 + 7 - 1$)	Inverses/complement of known additions facts (e.g., $12 - 5$ is known because $5 + 7 = 12$)
	Redistributed derived facts (e.g., $7 + 5 = 7 + (3 + 2) = (7 + 3) + 2 = 10 + 2 = 12$)	Redistributed derived facts (e.g., $12 - 5 = (7 + 5) - 5 = 7 + (5 - 5) = 7$)
Retrieval	Retrieval from long-term memory	Retrieval from long-term memory

Figure 10.1 The developmental process for basic fact mastery for addition and subtraction.

Source: Henry, V. J., & Brown, R. S. (2008). "First-Grade Basic Facts: An Investigation into Teaching and Learning of an Accelerated, High-Demand Memorization Standard." *Journal for Research in Mathematics Education*, 39(2), p. 156. Reprinted with permission.

Direct modeling by counting all or taking away

Levels	$8 + 6 = 14$	$14 - 8 = 6$
Level 1: Count all	Count All	Take Away
Level 2: Count on	Count On	To solve $14 - 8$ I count on $8 + ? = 14$ I took away 8 8 to 14 is 6 so $14 - 8 = 6$
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten $10 + 4$	$14 - 8$: I make a ten for $8 + ? = 14$ $8 + 2 + 4$ 6 $8 + 6 = 14$
Doubles $\pm n$	$6 + 8$ $= 6 + 6 + 2$ $= 12 + 2 = 14$	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011).
http://ime.math.arizona.edu/progressions/

Counting on

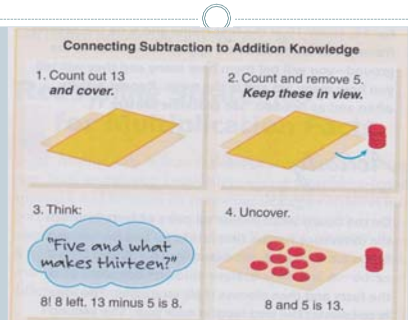
Adding (e. g., $8 + 6 = \square$) uses counting on to find a total: One counts on from the first addend (or the larger number is taken as the first addend). Counting on is monitored so that it stops when the second addend has been counted on. The last number word is the total.

Finding an unknown addend (e.g., $8 + \square = 14$): One counts on from the known addend. The keeping track method is monitored so that counting on stops when the known total has been reached. The keeping track method tells the unknown addend.

Subtracting ($14 - 8 = \square$): One thinks of subtracting as finding the unknown addend, as $8 + \square = 14$ and uses counting on to find an unknown addend (as above).

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Subtraction: Using a think-addition model



Van de Walle et al. (2010, p. 175)

Fact Mastery

- Develop efficient strategies for fact retrieval

Van de Walle et al. (2010)

Reasoning strategies - Addition

- Facts with Zero or Plus 10 (e.g., 4 + 0; 2 + 10)
- One More Than (e.g., 4 + 1) or Two More Than (e.g., 4 + 2)
- Turn arounds (e.g., 1 + 5 and 5 + 1)
- Doubles (e.g., 4 + 4)
- Doubles Plus One/Doubles Plus Two (e.g., 4 + 5; 4 + 6)
- Make Ten (e.g., 6 + 4; 8 + 2; 9 +1)
- Make Ten Extended (e.g., 7 + 5 = 7 + (3 +2) = (7 + 3) + 2) = 10 + 2 = 12)

Number/Fact Families

4 3 → 7

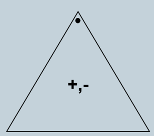
4 + 3 = 7 7 − 4 = 3

3 + 4 = 7 7 − 3 = 4

Addition/Subtraction Fact Families

- Start with a simple addition fact. (2 + 5 = 7)
- Change the addends around.
  (5 + 2 = 7)
- Write the inverse operations for both.
  (7 – 2 = 5 & 7 – 5 = 2)

Fact or Number Family –Addition and Subtraction



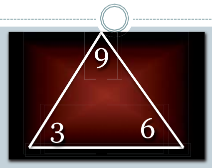
___ + ___ = ___

___ + ___ = ___

___ - ___ = ___

___ - ___ = ___

What are the members of this fact family?



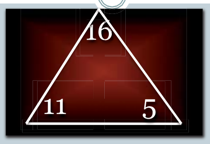
3 + 6 = 9

6 + 3 = 9

9 – 3 = 6

9 – 6 = 3

Extending thinking about fact family to other numbers?



11 + 5 = 16

5 + 11 = 16

16 – 11 = 5

16 – 5 = 9

When to Use Drill

- An efficient strategy for the skill to be drilled is already in place
- Automaticity with the skill or strategy is a desired outcome

Never use drill as a means of helping children learn!

Van de Walle (2007)

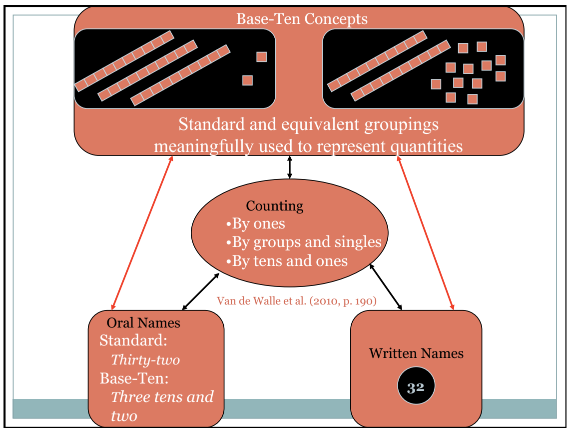
Computation

"Equally essential [with basic facts] is computational fluency – having and using efficient and accurate methods of computing. Fluency might be manifested in using a combination of mental strategies and jottings on paper or using an algorithm with paper and pencil, particularly when the numbers are large, to produce accurate results quickly" (NCTM, Principles and Standards, 2000, p. 32)

Understanding the base-ten system – place value and computation

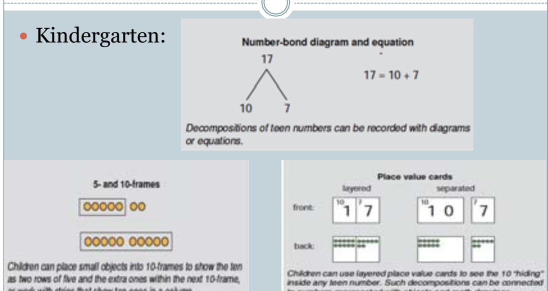
Relational understanding of basic place value requires an integration of new and difficult-to-construct concepts of grouping by tens (the base-ten concept) with procedural knowledge of how groups are recorded in our place-value scheme, how numbers are written, and how they are spoken.

Van de Walle et al. (2010)



Understanding the base-ten system

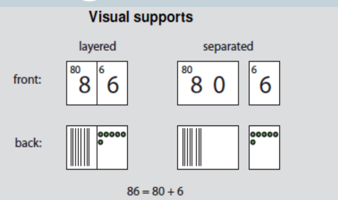
- Kindergarten:



Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Understanding place value

- Grade 1:



Drawings and place value cards can be used to connect number words and numerals to their base-ten meanings. Eighty-six is shown as "eight tens and six ones" and written as 86 and not as 806, a common Grade 1 error of writing what is heard.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Computation Algorithm

- "A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly" (p. 2)

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

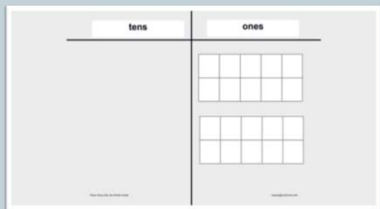
Explicit Instruction

- Expert Modeling
  - Use think-alouds and visual representations to demonstrate application of the strategy
- Guided Practice
  - Students practice applying concepts to solve problems with a partner or in small groups
  - Monitor student work and provide feedback
- Independent Work
  - Students generate information and solve problems on their own
  - Check work and provide feedback as needed

Addition algorithm : Models only

1. Display the problem at the top of the place value mat:

27
+54



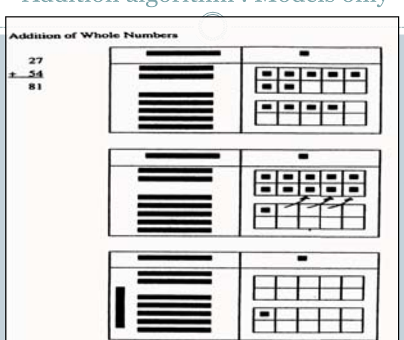
Addition algorithm : Models only

2. Think aloud as you attempt to answer the question.

Teacher: Here's one rule: *You begin in the ones column*. This is the way people came up with a long time ago, and it worked for them. Okay, I will start with the ones place. We have seven ones and 4 ones on our two ten-frames. I am going to fill the first ten-frames by moving some ones in the second ten-frame. *Can I make a trade?* Yes, I filled up 10. There's 11. That's 10 and an extra. So, I will trade the ten ones for a 10. I have one left in the ones column, which is not enough to trade tens. Now I will add the tens, which is 8 tens. The answer to 27 + 54 is 8 tens and a one – 81.

Van de Walle et al. (2010, p. 224)

Addition algorithm : Models only

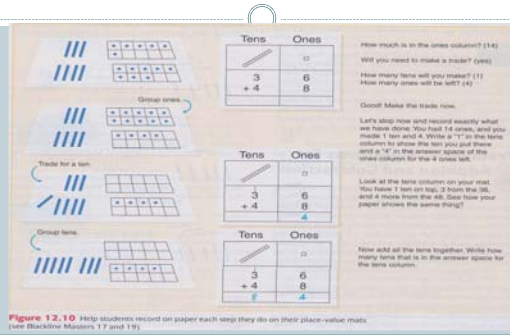


Can I make a trade?

Addition algorithm : Models only

3. Next, have students work through similar problems with you prompting them (ask leading questions) as needed.
4. Peer or partner group work or independent work with you monitoring it.
5. Finally, students do it independently and provide explanations.

Addition: Developing The Written Record



Van de Walle et al. (2010, p. 224)

Computation Strategies

- Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another
  - Special strategies
    - Example: 398 + 17 would mean decomposing 17 as 2 + 15 and evaluating (398 + 2) + 15
    - Cannot be extended to all numbers represented in the base-ten system or require considerable modification to do so.
    - When to use: In situations that require quick computation, but less so when uniformity is useful.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Grade 1

Special strategy: Counting on by tens

This strategy requires counting on by tens from 46, beginning 56, 66, 76, then counting on by ones.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

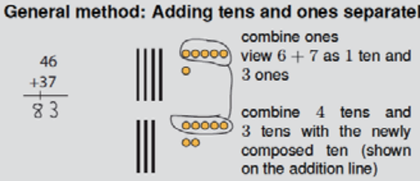
Computation Strategies

- General methods
  - Example: 398 +17 would require combining like base-ten units
  - Extend to all numbers in the base-ten system
  - Not the most efficient method (e.g., counting on by ones); however, methods based on place value are more efficient and closely connected with standard algorithms

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Grade 1

General method: Adding tens and ones separately



This method is an application of the associative property.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

- Addition Strategies from Investigations in number, data, and space

(Technical Education Research Center, 2008)

Addition Strategies

1a. Breaking the Numbers Apart – Adding by place (add hundreds, add tens, add ones, then combine)

349 + 175 =

300 + 100 = 400

40 + 70 = 110

9 + 5 = 14

400 + 110 +14 = 524

Addition Strategies

1b. Breaking the Numbers Apart – Adding on one number in parts (Add on hundreds, then tens, then ones)

349 + 175 =

349 + 100 = 449 449 + 70 = 519 519 + 5 = 524	349 + 100 = 449 449 + 50 = 499 499 + 25 = 524
349 + 50 = 399 399 + 50 = 449 449 + 50 = 499 499 + 25 = 524	

Addition Strategies

2a. Changing the Numbers – Changing to a landmark (use a nice number and compensate)

349 + 175 =

350 + 175 = 525 525 – 1 = 524	349 + 200 = 549 549 – 25 = 524

Addition Strategies

2b. Changing the Numbers – Creating an equivalent problem (move some to make hundreds)

349 + 175 =

324 + 200 = 524	400 + 124 = 524

(Technical Education Research Center, 2008).

Addition Strategies

3a. Adding more than two numbers –Breaking the numbers apart by adding by place

139 + 75 + 392 =

```
  139
   75
+ 392
  400   (100 + 300)
  190   (30 + 70 + 90)
   16   (9 + 5 + 2)
  606
```

Grade 2

Addition: Recording combined hundreds, tens, and ones on separate lines

```
  456      456      456      456
+ 167    + 167    + 167    + 167
-----    -----    -----    -----
           500      500      500
                    110      110
                              13
                             ---
                             623
```

Addition proceeds from left to right, but could also have gone from right to left. There are two advantages of working left to right: Many students prefer it because they read from left to right, and working first with the largest units yields a closer approximation earlier.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Grade 2

Addition: Recording newly composed units on the same line

```
  456      456      456      456
+ 167    + 167    + 167    + 167
-----      1       1 1      1 1
         -----    -----    -----
             3       23      623
```

Add the ones, 6 + 7, and record these 13 ones with 3 in the ones place and 1 on the line under the tens column.

Add the tens, 5 + 6 + 1, and record these 12 tens with 2 in the tens place and 1 on the line under the hundreds column.

Add the hundre[ds,] 4 + 1 + 1 and record these 6 hundreds in the hundreds colum[n].

Digits representing newly composed units are placed below the addends. This placement has several advantages. Each two-digit partial sum (e.g., "13") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 3) rather than "write the 3 and carry the 1" (write 3, then 1).

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Subtraction Algorithm – Models Only

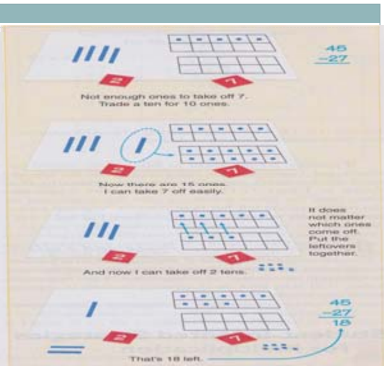
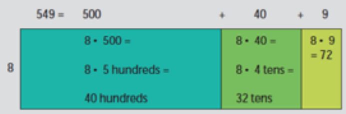


Figure 12.12 Two-place subtraction with models.

Van de Walle et al. (2010, p. 225)

Multiplication – Grade 4

Computation of 8×549 connected with an area model



Each part of the region above corresponds to one of the terms in the computation below.

$$8 \times 549 = 8 \times (500 + 40 + 9)$$
$$= 8 \times 500 + 8 \times 40 + 8 \times 9.$$

This can also viewed as finding how many objects are in 8 groups of 549 objects, by finding the cardinalities of 8 groups of 500, 8 groups of 40, and 8 groups of 9, then adding them.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Multiplication – Grade 4

Computation of 8×549: Ways to record general methods

Left to right showing the partial products

```
  549
 •  8     thinking:
 4000     8 • 5 hundreds
  320     8 • 4 tens
   72     8 • 9
 4392
```

Right to left showing the partial products

```
  549
 •  8     thinking:
   72     8 • 9
  320     8 • 4 tens
 4000     8 • 5 hundreds
 4392
```

Right to left recording the carries below

```
  549
 •  8
 4022
  3 7
 4392
```

The first method proceeds from left to right, and the others from right to left. In the third method, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from $8 \times 9 = 72$ is written diagonally to the left of the 2 rather than above the 4 in 549.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Multiplication

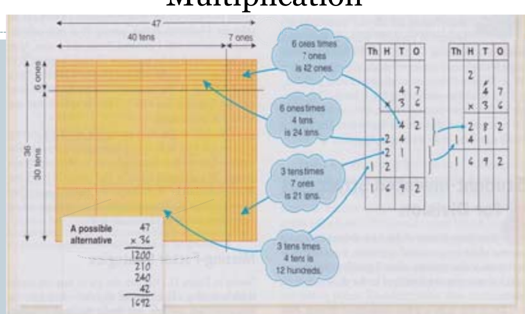
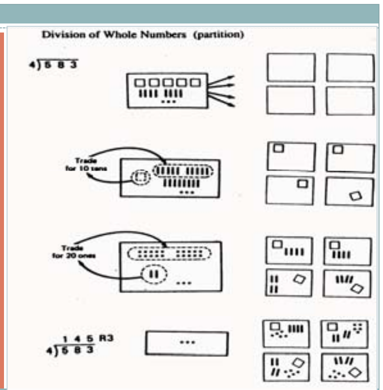


Figure 12.22 47 × 36 rectangle filled with base-ten pieces. Base-ten language connects the four partial

Van de Walle et al. (2010)

Division



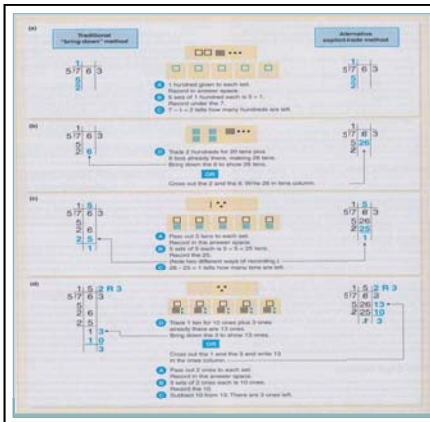
Develop the Written Record for Division: Partition or Fair Share Method

Help children learn to record the fair sharing with models. There are essentially four steps:

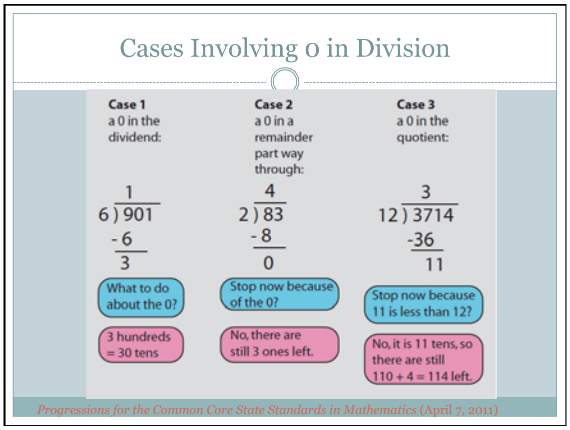
- ✓ *Share* and record the number of pieces put in each group.
- ✓ *Record* the number of pieces shared in all. Multiply to find this number.
- ✓ *Record* the number of pieces remaining. Subtract to find this number.
- ✓ *Trade* (if necessary) for smaller pieces, and combine with any that are there already. Record the new total number in the next column.

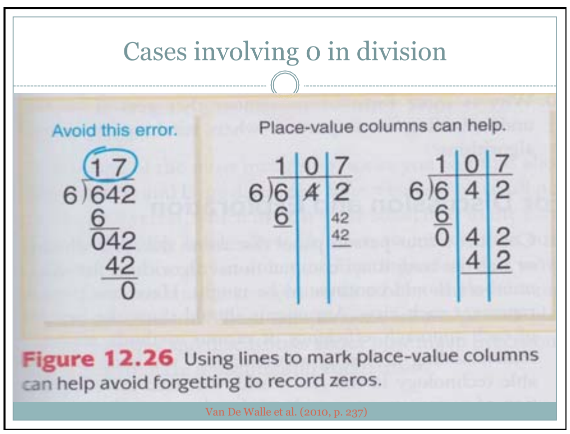
Van De Walle et al. (2010)

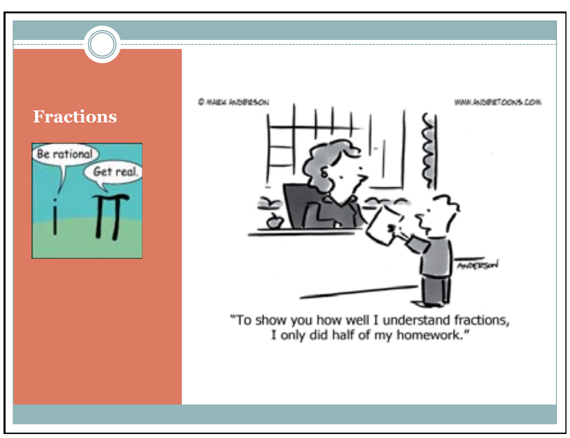
Division as finding group size



Van De Walle et al. (2010, p. 236)


Cases Involving 0 in Division
Case 1
a 0 in the dividend:
6) 901
1
-6
3
What to do about the 0?
3 hundreds = 30 tens
Case 2
a 0 in a remainder part way through:
2) 83
4
-8
0
Stop now because of the 0?
No, there are still 3 ones left.
Case 3
a 0 in the quotient:
12) 3714
3
-36
11
Stop now because 11 is less than 12?
No, it is 11 tens, so there are still 110 + 4 = 114 left.
Progressions for the Common Core State Standards in Mathematics (April 7, 2011)


Cases involving 0 in division
Avoid this error.
Place-value columns can help.
Figure 12.26 Using lines to mark place-value columns can help avoid forgetting to record zeros.
Van De Walle et al. (2010, p. 237)


Fractions
Be rational
Get real
"To show you how well I understand fractions, I only did half of my homework."

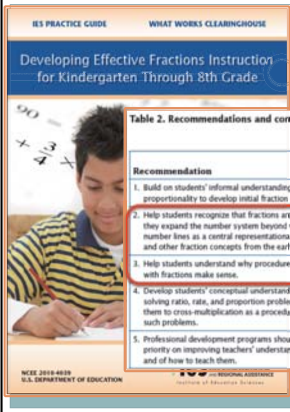


Table 2. Recommendations and corresponding levels of evidence

	Levels of Evidence		
Recommendation	Minimal Evidence	Moderate Evidence	Strong Evidence
1. Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.	◆		
2. Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.		◆	
3. Help students understand why procedures for computations with fractions make sense.		◆	
4. Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.	◆		
5. Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them.	◆		

Fractions

Begin fraction instruction with the prerequisite ability to:

- quickly and easily retrieve basic arithmetic facts
- execute arithmetic procedures involving whole numbers
- deeply understand core concepts involving whole numbers.

Note. Do not assume that children understand the magnitudes represented by fractions, even if they can perform arithmetic operations with them, or that children understand what the operations mean (e.g., understand what it means to multiply or divide one fraction by another).

National Mathematics Advisory Panel (2008)

Fraction Progressions for the CCSS in Mathematics

Grade 3
- The meaning of fractions
- The number line and number line diagrams
- Equivalent fractions
- Comparing fractions

Grade 4
- Equivalent fractions
- Adding and subtracting fractions
- Multiplication of fraction by a whole number
- Decimals

Grade 5
- Adding and subtracting fractions
- Multiplying and dividing fractions
- Multiplication as scaling

Meaning of Fractions

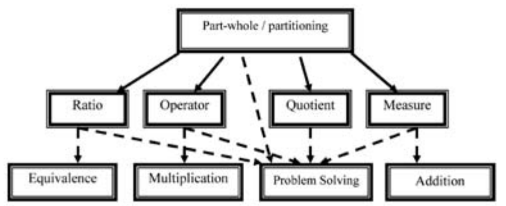


Figure 1. The theoretical model linking the five subconstructs of fractions to the different operations of fractions and to problem solving (Behr et al., 1983).

Development of Fraction Concepts

- Use problems represented in familiar contexts
- Emphasize many physical models and representations (even for older students) – pictorial, manipulative, verbal, real-world, and symbolic
- Develop the meanings of fractions

Reys, Lindquist, Lambdin, Smith, & Suydam (2004)

Models for Fractions

- Region or Area Models
  - Circular pie
  - Pattern blocks
  - Paper grids
- Length or Measurement Models
  - Fraction strips
  - Number lines
  - Folded paper strips
- Set Models
  - The whole is understood to be a set of objects. Subsets of the whole make up fractional parts

Van De Walle (2007)

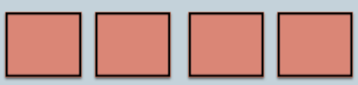
Meaning of Fractions: Where to Begin??

- Begin by helping children understand the idea of fractional parts of the whole
  - Sharing tasks
    - Initial strategies involve "halving." Good place to begin is with two, four, or eight sharers
- Introduce the vocabulary of fractional parts
  - The number of parts
  - The equality of the parts (in size, not shape)

Van De Walle (2004)

Sharing Tasks

Two children are sharing 4 brownies so that each one will get the same amount. How much can each child have?

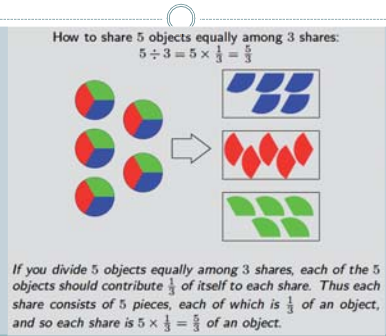


Now try this one…

7 pizzas with 6 children

Van De Walle (2004)

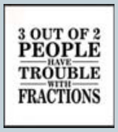
Division of fractions: Sharing



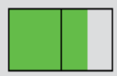
Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Meaning of Fractions–Grade 3

- Specifying the whole
- Explaining what is meant by "equal parts"



The importance of specifying the whole



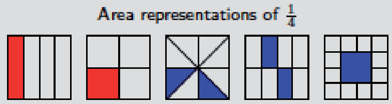
Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$; if the entire rectangle is the whole, the shaded area represents $\frac{3}{4}$.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Meaning of Fractions–Grade 3

- "Equal parts" as parts with equal measurement

Area representations of $\frac{1}{4}$

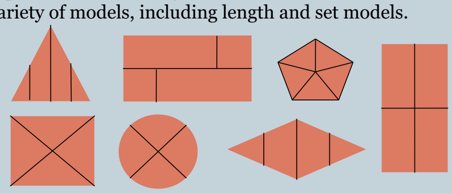


In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

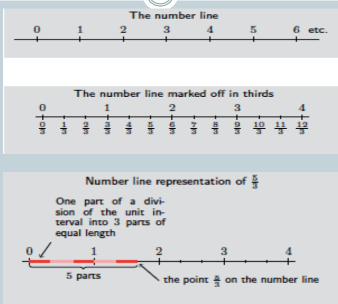
Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Activity: Correct Shares

Show examples and nonexamples of specified fractional parts (e.g., one-fourths). Have students identify the wholes that are correctly divided into requested fractional parts and those that are not. For each response, have students explain their reasoning. The activity should be done with a variety of models, including length and set models.

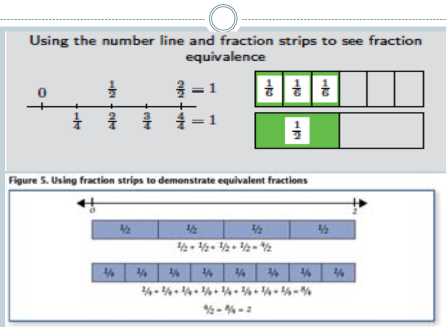


Number line and number line diagrams – Grade 3



Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Equivalent fractions using the number line and fraction strips– Grade 3



Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

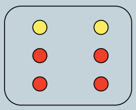
An example of explicit instruction: Equivalent fractions

1. Display the problem: Generate an equivalent fraction for $\frac{4}{6}$.
2. Think aloud as you attempt to answer the question. It is important that you model your thinking rather than tell your students how to do the problem.

$$\frac{4}{6} = \frac{2}{3}$$

Equivalent Fractions

Teacher: Ok, I think I will use models to help me and start with the fraction $\frac{4}{6}$. If I have a set of 6 things and I take 4 of them, then that would be $\frac{4}{6}$.

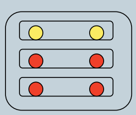


Now I need to show how this $\frac{4}{6} = \frac{2}{3}$.

Equivalent Fractions

Teacher:

I can make the 6 into groups of 2. So then there would be 3 groups, and the 4 would be 2 groups out of the 3 groups. That means it is $\frac{2}{3}$.



Equivalent Fractions

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Teacher: Another way to think about $\frac{4}{6}=\frac{2}{3}$ is to draw a picture and start with the $\frac{2}{3}$. I will draw a square cut into 3 parts and shade in 2, then that would be $\frac{2}{3}$ shaded.

Equivalent Fractions

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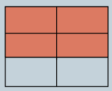
Teacher:

Now I need to show how this $\frac{2}{3} = \frac{4}{6}$. If I cut all 3 of these parts in half, that would be 4 parts shaded and 6 parts in all. That's $\frac{4}{6}$, and it would be the same amount.



Equivalent Fractions

3. Next, have students work through similar problems with you prompting them (asking leading questions) as needed.
4. Then have students do peer or partner group work or independent work with you monitoring it.
5. Finally, students should be able to do it independently and provide explanations.

Remember: Students should learn to develop an understanding of equivalent fractions and also develop from that understanding a conceptually based algorithm

Van de Walle (2007)

Missing-Number Equivalencies

$\frac{2}{3} = \frac{8}{\_\_}$

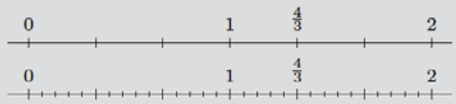
$\frac{4}{3} = \frac{\_\_}{15}$

$\frac{8}{12} = \frac{\_\_}{3}$

$\frac{9}{12} = \frac{3}{\_\_}$

Equivalent fractions using a number line

Using the number line to show that $\frac{4}{3} = \frac{5\times4}{5\times3}$

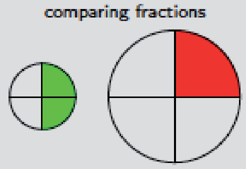


$\frac{4}{3}$ is 4 parts when each part is $\frac{1}{3}$, and we want to see that this is also 5×4 parts when each part is $\frac{1}{5\times3}$. Divide each of the intervals of length $\frac{1}{3}$ into 5 parts of equal length. There are 5×3 parts of equal length in the unit interval, and $\frac{4}{3}$ is 5×4 of these. Therefore $\frac{4}{3} = \frac{5\times4}{5\times3} = \frac{20}{15}$.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Comparing Fractions: Grade 3

The importance of referring to the same whole when comparing fractions



A student might think that $\frac{1}{4} > \frac{1}{2}$, because a fourth of the pizza on the right is bigger than a half of the pizza on the left.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Compare fractions

Order these fractions without trying to find a common denominator:

$$\frac{13}{25}, \frac{2}{31}, \frac{5}{6}, \frac{4}{11}, \text{and } \frac{21}{20}$$

Van de Walle, Karp, & Bay-Williams, 2010

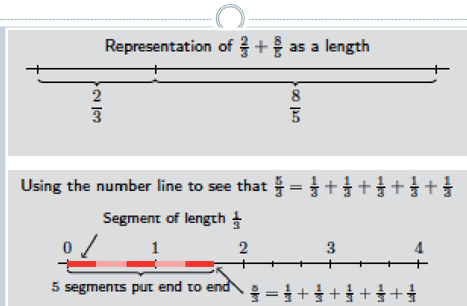
Comparing fractions

Which fraction in each pair is greater?
Give one or more reasons. Try not to use drawings or model
Do not use common denominators or cross-multiplication.
Rely on concepts.

- A. $\frac{4}{5}$ or $\frac{4}{9}$
- B. $\frac{4}{7}$ or $\frac{5}{7}$
- C. $\frac{3}{8}$ or $\frac{4}{10}$
- D. $\frac{5}{3}$ or $\frac{5}{8}$
- E. $\frac{3}{4}$ or $\frac{9}{10}$
- F. $\frac{3}{8}$ or $\frac{4}{7}$
- G. $\frac{7}{12}$ or $\frac{5}{12}$
- H. $\frac{3}{5}$ or $\frac{3}{7}$
- I. $\frac{5}{8}$ or $\frac{6}{10}$
- J. $\frac{9}{8}$ or $\frac{4}{3}$
- K. $\frac{4}{6}$ or $\frac{7}{12}$
- L. $\frac{8}{9}$ or $\frac{7}{8}$

Van de Walle, Karp, & Bay-Williams, 2010

Adding fractions – Grade 4



Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Subtracting fractions with fraction bars

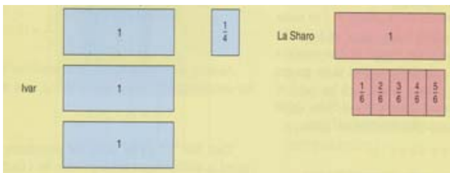
Problem: Ivar has $3\frac{1}{4}$ candy bars. La Sharo has $1\frac{5}{6}$ candy bars. How much more candy does Ivar have than La Sharo?

Reys, Lindquist, Lambdin, Smith, & Suydam (2004, p. 299)

Subtracting fractions with fraction bars

Problem: Ivar has $3\frac{1}{4}$ candy bars. La Sharo has $1\frac{5}{6}$ candy bars. How much more candy does Ivar have than La Sharo?

1) Model the problem with fraction bars.

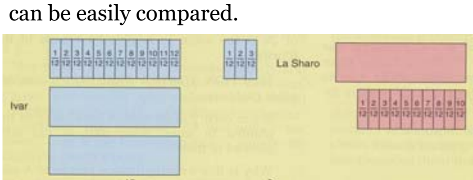


2) Estimate the problem to the nearest whole number (approximately 3 – 2= 1)

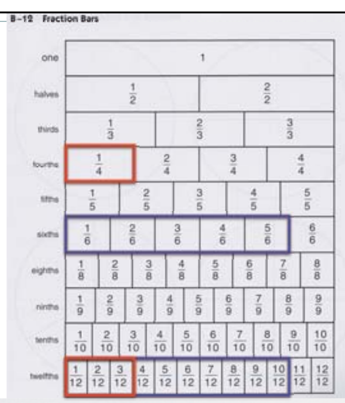
Reys, Lindquist, Lambdin, Smith, & Suydam (2004, p. 299)

Subtracting fractions with fraction bars

3) Trade the strips until the two sets of candy bars can be easily compared.



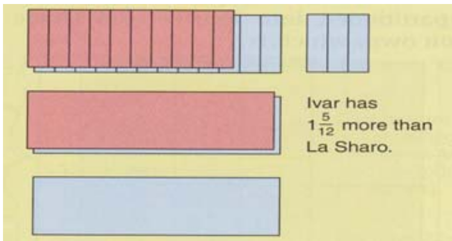
Reys, Lindquist, Lambdin, Smith, & Suydam (2004, p. 299)



Reys et al., *Helping Children Learn Mathematics*, 7E, copyright © 2004 by John Wiley & Sons, Inc.

Subtracting fractions with fraction bars

4) Compare La Sharo's $1\frac{10}{12}$ candy bars with Ivar's $3\frac{3}{12}$ candy bars.



Reys, Lindquist, Lambdin, Smith, & Suydam (2004, p. 299)

Subtracting fractions with fraction bars

5) Show a symbolic representation

$$\begin{array}{r} 3\frac{1}{4} \\ -1\frac{5}{6} \\ \hline \end{array} \rightarrow \begin{array}{r} 3\frac{3}{12} \\ -1\frac{10}{12} \\ \hline \end{array} \rightarrow \begin{array}{r} 2\frac{15}{12} \\ -1\frac{10}{12} \\ \hline 1\frac{5}{12} \end{array}$$

Reys, Lindquist, Lambdin, Smith, & Suydam (2004, p. 299)

Multiplying with Fractions

If you own $\frac{3}{4}$ of an acre of land and $\frac{5}{6}$ of this is planted in trees, what part of the acre is planted in trees.

Reys, Lindquist, Lambdin, Smith, & Suydam (2004)

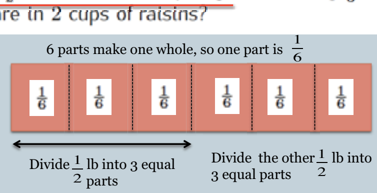
Story problems to make sense of division- Grade 5

How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Story problems to make sense of division- Grade 5

How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?



Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

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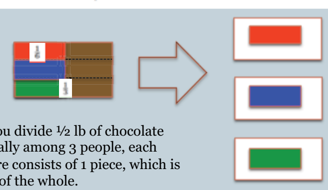
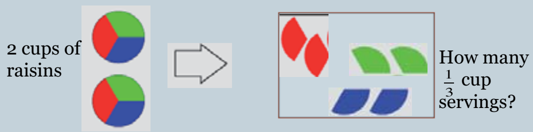


If you divide ½ lb of chocolate equally among 3 people, each share consists of 1 piece, which is 1/6 of the whole.

Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Story problems to make sense of division- Grade 5

How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?



Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

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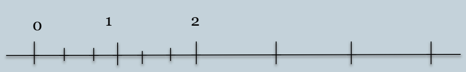


Progressions for the Common Core State Standards in Mathematics (April 7, 2011)

Summary

USE EXPLICIT, SYSTEMATIC INSTRUCTION TO TEACH STUDENTS WITH MATH DIFFICULTIES

- PROVIDE MODELS OF PROFICIENT PROBLEM SOLVING
- VERBALIZE THOUGHT PROCESSES
- GUIDED PRACTICE
- CORRECTIVE FEEDBACK
- FREQUENT CUMULATIVE REVIEW

Thanks!

Asha K. Jitendra (jiten001@umn.edu)

RATIO, *PROPORTION*, and *PERCENT* PROBLEM CHECKLISTS

Step 1: Discover the problem type

- Read and retell the problem to understand it.
- Ask if the problem is a …

RATIO PROBLEM (a : b)	PROPORTION PROBLEM (*If…Then*)	PERCENT, PERCENT OF CHANGE, or SIMPLE INTEREST PROBLEM (%)
Does this problem have a part-to-part or part-to-whole comparison? Look for symbols, words, and phrases such as: "the ratio of ***a*** to ***b***," "***a*** : ***b***," "***a*** per ***b***," "***a*** for ***b***," "***a*** for every ***b***," "for every ***b*** there are ***a***," "***n*** times as many/much as," "***n***th of," "***a*** out of ***b***," to see whether there is a **ratio statement** that tells about a multiplicative relationship between two quantities in a single situation.	Does the problem describe an ***"If…Then"*** statement of equality between two ratios/rates that allows us to think about the ways that two situations are the same? That is, the ***If*** statement describes a rate/ratio between two quantities in one situation and the ***Then*** statement involves either an increase or decrease in the two quantities in another situation, but with the same ratio.	Look for symbols or words such as **"%," "percent," "percent of change,"** or **"simple interest,"** to see whether there is a **percent or percent of change statement** that tells about a multiplicative relationship between two quantities.

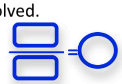
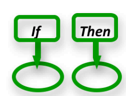
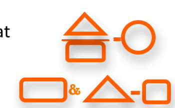
- Ask if this problem is different from or similar to another problem that has already been solved.

Step 2: Identify information in the problem to represent in a diagram(s)

- Underline the …

ratio or comparison statement.	two quantities that form a specific ratio/rate.	percent or simple interest statement.

- Write …

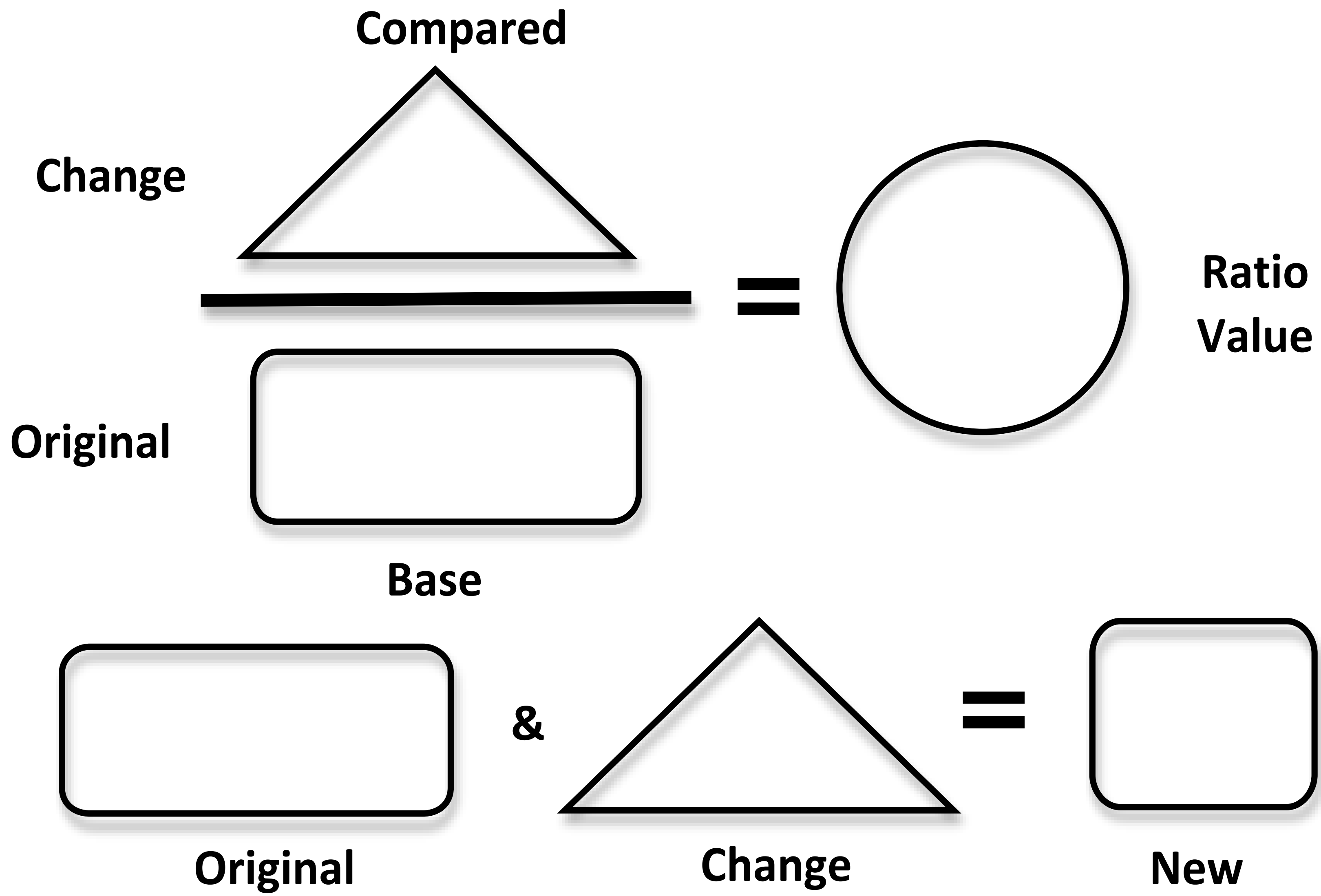
→ compared and base quantities and units in the diagram. → value of the ratio between the two quantities in the diagram (**ratio value**). → "***x***" for what must be solved. 	→ names of the two quantities that form a ratio in the diagram. → quantities and units for each of the two ratios/rates in the diagram. "***x***" for what must be 	→ information (part, whole, or ratio value; change, original, ratio value, or new) in the problem in the diagram(s). → "***x***" for what must be solved. 

Step 3: Solve the problem

- Try to come up with an estimate for the answer.
- Translate the information in the diagram into a math equation.
- Plan how to solve the math equation.
- Solve the math equation, and write the complete answer.

Step 4: Check the solution

- Look back to see if the estimate in Step 3 is close to the exact answer.
- Check to see if the answer makes sense.


Compared
Change
Ratio
Value
Original
Base
&
Original
Change
New